A Gentle Introduction To Classical Field Theory

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Fields are ubiquitous in modern physics. From classical gravitation and electromagnetism to inflationary expansion in the early universe, they explain a wide range of phenomena. In this pedagogical introduction, we cover the motivations behind the introduction of *classical* fields, going from discrete systems to continuous ones, and then derive the Lagrangian equations of motion for a field, finally deriving the generalised Noether's theorem. Finally, we use Noether's theorem to see the significance of the energy-momentum tensor of a field.

1. MOTIVATIONS AND DEFINITIONS

Although technically, Newton's Law of Gravitation, which ascribes an expression for the strength of the attractive force between masses at each point is a field theory, the term itself was first used by Michael Faraday in 1849, and the early workers of electrodynamics first found it convenient to describe the interactions of charges and currents through the field picture.

With Maxwell's discovery that light propagates at a finite speed, and the setting up of source-effect equations for electric and magnetic fields like the one Newton had formulated for gravity, the field notion gained credence, to explain this 'action at a distance', the effects of sources on objects not in contact. A source would set up a region of influence around it, which acted on all other elements present in this field.

Apart from explaining the 'action of matter where it is not', the idea of a field helps extend mechanics to a wider range of systems. In preliminary treatments of classical mechanics, the Lagrangian and Hamiltonian formalisms are used to describe the motion of a finite, countable number of freedoms, which can easily be used to cover systems with countable, infinite degrees of freedom. The picture of a field covers systems which are *continuous* and infinite.

A **classical field**, then, is a quantity $\phi(x,t)$ given to each point in space at a point in time. If the coordinates of an object are represented at $q_i(t)$ in the case of discrete systems, $\phi(x,t)$ replaces q and x is the continuous 'label' which replaces the subscript i. Common examples of classical fields include the the electric, magnetic and gravitational fields.

2. FROM CHAIN TO STRING

The simplest illustrative example of a field is the picture of a string, which can be visualised as a chain of oscillators, that tends towards a continuous system in the limit of very small distances between each oscillating unit.

We consider a system made up of N particles, each of mass m, each connected to the next by a massless spring of spring constant k. The position of each particle is denoted by q_i, and the particles can only move along the length of this chain-like system.

We will first use the Langrangian picture to study this arrangement, as the motions of a particle on a spring is conveniently described by the same. Also, in further extensions to quantum field theory, the Lagrangian density is a centrally important entity constructed from the most general symmetries of the system.

Coming back to our picture of the long chain of masses, the total kinetic energy can be written as:

$$T = \sum_i^N \frac{1}{2} m\dot{q}_i^2 \tag{1}$$

And the total potential energy is:

$$V = \sum_i^N \frac{1}{2} k(q_{i+1} - q_i)^2 \tag{2}$$

So the Lagrangian, as usual, becomes

$$L = T - V = \sum_i^N \frac{1}{2} m\dot{q}_i^2 - \frac{1}{2} k(q_{i+1} - q_i)^2 \tag{3}$$

From this, the equation of motion for the particle q_i becomes:

$$m\ddot{q}_i = k(q_{i+1} - q_i) - k(q_i - q_{i-1}) \tag{4}$$

We can rewrite (4) (where l is the natural length of each spring):

$$\frac{m}{l}\ddot{q}_i = \frac{kl(q_{i+1} - q_i)}{l^2} - \frac{kl(q_i - q_{i-1})}{l^2} \tag{5}$$

Now, taking $N \to \infty$ so that $a \to 0$ and $m \to 0$, so that the chain tends towards a continuous system, we see the expression m/l reduces to the mass per unit length ,

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denoted by μ, and ka becomes κ, the Young's Modulus of the chain (now string).

Also, in jumping from discrete to continuous notation, we denote the label of the ith particle by $x = il$, so that, in the new notation, $q_i \to \phi(x)$, and the expression of the difference of two mass points tends to the derivative at that point:

$$\lim_{l\to 0} \frac{kl(q_{i+1}-q_i)}{l} = \lim_{l\to 0} \frac{k[\phi((i+1)l)-\phi(il)]}{l} = \frac{\partial\phi}{\partial x}$$

The right hand side of (5) can be then written as:

$$\lim_{l\to 0} \frac{\kappa}{l}\left[\left.\frac{\partial\phi}{\partial x}\right|_x - \left.\frac{\partial\phi}{\partial x}\right|_{x-l}\right] = \kappa\frac{\partial^2\phi}{\partial x^2}$$

The final expression of (5), in our continuous limit is:

$$\mu\frac{\partial^2\phi}{\partial t^2} = \kappa\frac{\partial^2\phi}{\partial x^2} \tag{6}$$

The Lagrangian becomes:

$$L = \int \frac{1}{2}\left[\mu\left(\frac{\partial\phi}{\partial t}\right)^2 - \kappa\left(\frac{\partial\phi}{\partial x}\right)^2\right]dx = \int \mathcal{L}dx \tag{7}$$

So that the *Lagrangian density* is given by:

$$\mathcal{L} = \frac{1}{2}\left[\mu\left(\frac{\partial\phi}{\partial t}\right)^2 - \kappa\left(\frac{\partial\phi}{\partial x}\right)^2\right] \tag{8}$$

We can see that (6) is the common wave equation, which we have successfully derived by considering the chain as a continuous system. This constitutes a rudimentary field theory, specifying the displacement of the string, ϕ at each point x along the string, and specifying its dynamics.

3. THE EULER-LAGRANGE EQUATIONS FOR A CLASSICAL FIELD

Since we have already seen the utility of the Lagrangian formulation of mechanics in describing the simplest field, it is only natural to try and extend the machinery of the Euler-Lagrange equations to compute field dynamics. Knowledge of the Lagrangian density of a given field will then be enough to describe its behaviour at all points and times.

From here on, we shall use the Einstein summation convention, where repeated indices are summed over, and also use ∂_μ to mean a derivative with respect to the component x^μ, where Greek indices go from 0 to 3, denoting time and three space components.

We will consider the Lagrangian density as a function of only the field, its first derivatives in space and time, and possibly time: $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi, t)$. It is understood that any dependence on space coordinates is implicit in the dependence on the field itself. We consider the action of the field and perturb it:

$$S = \int d^3xdt\mathcal{L} \tag{9}$$

$$\delta S = \int d^3xdt\left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right] \tag{10}$$

Applying integration by parts, we have:

$$\delta S = \int d^3xdt\left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)\right]$$

Since we are integrating over all space, and we assume that any real field decays to zero at infinity, the third term drops to zero on integration. Now, since the variation of the action under our perturbation is zero, we impose $\delta S = 0$, which means the remaining integrand is zero. This gives us the Euler-Lagrange Equations for a field:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \frac{\partial\mathcal{L}}{\partial\phi} \tag{11}$$

4. THE HAMILTONIAN PICTURE

The Poisson bracket formalism of classical mechanics and the canonical quantisation relations of quantum mechanics provide a bridge between the two realms, and the path from classical to quantum field theory goes through an analogous elevation of the field to an observable, and the establishment of commutation relations between the field and a suitably conjugate momentum. It is thus instructive to briefly cover the Hamiltonian way of looking at fields, as an aside.

We define the momentum conjugate to a field as the derivative of the Lagrangian with respect to the time derivative of the field:

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} \tag{12}$$

Like the case with ordinary point particles, we perform a Legendre transform on the Lagrangian to get to the Hamiltonian, with the defined conjugate momentum as the changing variable. Thus, the Hamiltonian becomes:

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} \tag{13}$$

For instance, if the field has the simple Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)$$

The Hamiltonian becomes:

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 + V(\phi) \qquad (14)$$

5. SYMMETRIES AND CONSERVATION

Perhaps the most beautiful theorem in all of mechanics is that named after and proven by Emmy Noether: every symmetry of the system has associated with it a conserved quantity. Here, we will consider a general symmetry transformation, and derive an expression for a conserved current, thereby extending Noether's Theorem to classical fields.

Let us consider a symmetry transformation, where the change in the field leaves the motion invariant. The transformation where $\phi \to \phi + \delta\phi$ and $\delta\phi = f(\phi)$. If this leaves the motion unchanged, we can say that the field Lagrangian changes by a total derivative:

$$\mathcal{L} \to \mathcal{L} + \partial_\nu F^\nu$$
$$\delta\mathcal{L} = \partial_\nu F^\nu$$

Now, we consider a general perturbation of the Lagrangian under this transformation:

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) \\ &= \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)\end{aligned}$$

Upon using the Euler-Lagrange Equations (11), the first two terms cancel out, and we are left with:

$$\delta\mathcal{L} = \partial_\mu F^\mu = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)$$
$$\Rightarrow \partial_\mu\left(F^\mu - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) = 0$$
$$\Rightarrow \partial_\mu\left(F^\mu - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}f(\phi)\right) = 0$$

We clearly see a conserved quantity associated with this symmetry. We define it to be the *conserved current* J^μ, and thus the conservation expression becomes:

$$\partial_\mu J^\mu = 0 \qquad (15)$$

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}f(\phi) - F^\mu \qquad (16)$$

Expanding (15) leads to a continuity equation:

$$\frac{\partial}{\partial t}J^0 + \partial_i J^i = 0 \qquad (17)$$

Where J^0 acts as an effective charge and the three-vector J^i as an effective outgoing current, signifying that the charge is locally conserved. Upon integrating (17) over all space, the outgoing current terms drop out to zero at infinity, and this means:

$$\frac{\partial}{\partial t}\int d^3x J^0 = \frac{\partial Q}{\partial t} = 0 \qquad (18)$$

Which means the charge Q is conserved globally. This means each conserved current has with it a globally conserved charge.

6. TRANSLATIONAL SYMMETRY AND ENERGY-MOMENTUM

To illustrate Noether's theorem, we will use a simple symmetry transformation, that of space and time translation, so that $x^\nu \to x^\nu + \epsilon^\nu$. The $\nu = 0$ component of this transformation leads to time translation, and the $\nu = 1, 2, 3$ components mean space translation. Now, under our prescribed transformation, we can say that the field and the Lagrangian transform as:

$$\phi(x) \to \phi(x) + \epsilon^\nu\partial_\nu\phi \qquad (19)$$
$$\mathcal{L} \to \mathcal{L} + \epsilon^\nu\partial_\nu\mathcal{L} \qquad (20)$$

This transformation can be thought of as a combination of four separate translations, one in each dimension. Therefore, for each one, we get a four-vector of conserved current.

For instance, for $\nu = 0$, we have $\delta\phi = \partial_0(\epsilon^0\phi)$. From (16), we can write (ϵ cancels out):

$$(J^\mu)_0 = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_0\phi - \delta^\mu_0\mathcal{L} \qquad (21)$$

Extending this to the other three components, we can generally write:

$$(J^\mu)_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L} \qquad (22)$$

Looking at the quantity $(J^\mu)_\nu$, we can see it has 16 components, and thus we denote it in tensor notation:

$T^{\mu}_{\ \nu}$. This tensor has special significance, and we shall proceed to uncover it thus. In accordance with (18), we know that for each index ν, there will be a conserved charge, which will have four components:

$$Q^{\nu} = \int d^3x\, T^{0\nu} \tag{23}$$

First, let us look at the case $\nu = 0$. Here, $Q^0 = \int d^3x\, T^{00}$ and from (22), $T^{00} = \pi\dot{\phi} - \mathcal{L}$, where π is the conjugate momentum. We see that

$$E = \int d^3x\, (\pi\dot{\phi} - \mathcal{L})$$

is conserved. From the definition of the Hamiltonian density of the field (13), this is just the Hamiltonian, and the conserved quantity then turns out to be just the energy of the system. Remember, this calculation was for the $\nu = 0$, i.e. time component of the transformation. Thus, we see that the imposition of invariance under time translation has automatically implied the conservation of energy!

Similarly, the conservation of charges for the cases $\nu = 1, 2, 3 = i$, leads to the following quantities being conserved:

$$P_i = \int d^3x\, \pi\partial_i\phi$$

P_i is just the ith component of total field momentum. Thus, again, we see how invariance under space translations leaves the total momentum conserved!

Since the tensor $T^{\mu}_{\ \nu}$ has led to the physical realisation of the conservation of momentum and energy, it is called the **energy-momentum tensor**.

General References

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